The Student's Essential Formula Book

Mathematical Formulas, Tables, Puzzles and Curios

1st Edition

Compilation by John C. Sparks

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ISBN: X-XX-XXXX-X

First Edition May 2005

Published by:

AUTHOR HOUSE PUBLISHING

Bloomington, Indiana

Forward

Formulas, they seem to be the bane of every beginning mathematics student who has yet to realize that formulas are about structure and relationship—and not about memorization. Granted, formulas have to be memorized; for, it is partly through memorization that we eventually become ‘unconsciously competent’. This means we are a true master of our skill, practicing it in an almost effortless, automatic sense. In mathematics, this means we have mastered the underlying algebraic language to the same degree that we have mastered our native tongue. Knowing formulas and understanding the reasoning behind them propels one towards the road to mathematical mastery, so essential in our modern high-tech society.

This book consists of three major sections. Section I—Formulas—contains most of the formulas that you would expect to encounter through the first year of college (and perhaps the second) regardless of major. In addition, there are formulas rarely seen in such compilations, included as a mathematical treat for the inquisitive. Section II—Tables—includes both ‘pure math’ tables and physical-science tables, useful in a variety of disciples ranging from physics to nursing. As in Section I, some tables are included just to nurture curiosity in a spirit of fun. *Fun in discovery definitely should be a part of our learning experience in mathematics.* Section III—Puzzles and Curios—is all fun! Here, I have pulled together a variety of mathematical wonders and puzzles collected over three decades of teaching.

I would like to thank Mr. Al Giambrone—Chairman of the Department of Mathematics, Sinclair Community College, Dayton, Ohio—for providing required-memorization formula lists for 22 Sinclair mathematics courses from which the formula compilation was partially built. In addition, I would like to thank Mr. Vince Miller—Adjunct Professor, Department of Mathematics, Sinclair Community College—for providing a careful and exacting review of the final manuscript.

John C. Sparks
December 2004

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Dedication

I would like to dedicate this work
to my wife and family—
Carolyn, Robert, and Curtis Sparks.
Also… to the Ultimate Mathematician.

Significance

The wisp in my glass on a clear winter's night
Is home for a billion wee glimmers of light,
Each crystal itself one faraway dream
With faraway worlds surrounding its gleam.

And locked in the realm of each tiny sphere
Is all that is met through an eye or an ear;
Too, all that is felt by a hand or our love,
For we are but whits in the sea seen above.

Such scales immense make wonder abound
And make a lone knee touch the cold ground.
For what is this man that he should be made
To sing to The One whose breath heavens laid?

July 1999

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Part I

Formulas

1) Algebra

1.1 Field Axioms

The field axioms *decree* the fundamental operating properties of the real number system and provide the basis for all advanced operating properties in mathematics.

Let $a, b \,\&\, c$ be any three real numbers		
Properties	**Addition**	**Multiplication**
Closure	$a+b$ is a unique real number	$a \cdot b$ is a unique real number
Commutative	$a+b=b+a$	$a \cdot b = b \cdot a$
Associative	$(a+b)+c=$ $a+(b+c)$	$(ab)c=$ $a(bc)$
Identity	$0 \Rightarrow a+0=a$	$1 \Rightarrow a \cdot 1 = a$
Inverse	$a \Rightarrow a+(-a)=0$ $\Rightarrow (-a)+a=0$	$a \neq 0 \Rightarrow a \cdot \frac{1}{a} = 1$ $\Rightarrow \frac{1}{a} \cdot a = 1$
Distributive *or Linking Property*	$a \cdot (b+c) = a \cdot b + a \cdot c$	
Note: $ab = a(b) = (a)b$ are alternate representations of $a \cdot b$		

1.2 Divisibility Tests

DIVISOR	CONDITION THAT MAKES IT SO
2	The last digit is 0,2,4,6, or 8
3	The sum of the digits is divisible by 3
4	The last two digits are divisible by 4
5	The last digit is 0 or 5
6	The number is divisible by both 2 and 3
7	The number formed by adding five times the last digit to the remaining digits is divisible by 7**
8	The last three digits are divisible by 8
9	The sum of the digits is divisible by 9
10	The last digit is 0
11	11 divides the number formed by subtracting two times the last digit from the remaining digits**
12	The number is divisible by both 3 and 4
13	13 divides the number formed by adding four times the last digit to the remaining digits**
14	The number is divisible by both 2 and 7
15	The number is divisible by both 3 and 5
17	17 divides the number formed by subtracting five times the last digit from the remaining digits**
19	19 divides the number formed by adding two times the last digit to the remaining digits**
23	23 divides the number formed by adding seven times the last digit to the remaining digits**
29	29 divides the number formed by adding three times the last digit to the remaining digits**
31	31 divides the number formed by subtracting three times the last digit from the remaining digits**
37	37 divides the number formed by subtracting eleven times the last digit from the remaining digits**
**These tests are iterative tests in that you continue to cycle through the process until a number is formed that can be easily divided by the divisor in question.	

1.3 Subtraction and Division

1. Definitions

   Subtraction: $a - b \equiv a + (-b)$

   Division: $a \div b \equiv a \cdot \dfrac{1}{b}$

2. Alternate representation of $a \div b$: $a \div b = \dfrac{a}{b}$

3. Division Properties of Zero

   Zero in numerator: $a \neq 0 \Rightarrow \dfrac{0}{a} = 0$

   Zero in denominator: $\dfrac{a}{0}$ is undefined!

   Zero in both: $\dfrac{0}{0}$ is undefined!

1.4 Rules for Fractions

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be fractions with $b \neq 0$ and $d \neq 0$.

1. Equality: $\dfrac{a}{b} = \dfrac{c}{d} \Leftrightarrow ad = bc$

2. Equivalency: $c \neq 0 \Rightarrow \dfrac{a}{b} = \dfrac{ac}{bc} = \dfrac{ca}{cb} = \dfrac{ac}{cb} = \dfrac{ca}{bc}$

3. Addition (like denominators): $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a+c}{b}$

4. Addition (unlike denominators):

   $$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$$

5. Subtraction (like denominators): $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$

6. Subtraction (unlike denominators):

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{cb}{bd}=\frac{ad-cb}{bd}$$

7. Multiplication: $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$

8. Division: $c \neq 0 \Rightarrow \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$

9. Reduction of Complex Fraction: $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}$

10. Placement of Sign: $-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$

1.5 Rules for Exponents

1. Addition: $a^n a^m = a^{n+m}$
2. Subtraction: $\frac{a^n}{a^m}=a^{n-m}$
3. Multiplication: $(a^n)^m = a^{nm}$
4. Distributed over a Simple Product: $(ab)^n = a^n b^n$
5. Distributed over a Complex Product: $(a^m b^p)^n = a^{mn} b^{pn}$
6. Distributed over a Simple Quotient: $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$
7. Distributed over a Complex Quotient: $\left(\frac{a^m}{b^p}\right)^n=\frac{a^{mn}}{b^{pn}}$

8. Definition of Negative Exponent: $\frac{1}{a^n} \equiv a^{-n}$

9. Definition of Radical Expression: $\sqrt[n]{a} \equiv a^{\frac{1}{n}}$

10. Definition when No Exponent is Present: $a = a^1$

11. Definition of Zero Exponent: $a^0 = 1$

1.6 Factor Formulas

1. Simple Common Factor: $ab + ac = a(b + c) = (b + c)a$
2. Grouped Common Factor:
$$ab + ac + db + dc = (b + c)a + d(b + c) =$$
$$(b + c)a + (b + c)d = (b + c)(a + d)$$
3. Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$
4. Sum of Squares: $a^2 + b^2 = (a + bi)(a - bi)$
5. Perfect Square: $a^2 \pm 2ab + b^2 = (a \pm b)^2$
6. General Trinomial: $x^2 + (a + b)x + ab = (x + a)(x + b)$
7. Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
8. Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
9. Power Reduction to an Integer:
$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$$
10. Power Reduction to a Radical: $x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$
11. Power Reduction to an Integer plus a Radical:
$$a^2 + ab + b^2 = (a + \sqrt{ab} + b)(a - \sqrt{ab} + b)$$

1.7 Laws of Equality

Let $A = B$ be an algebraic equality and C, D be any quantities.

1. Addition: $A + C = B + C$

2. Subtraction: $A - C = B - C$
3. Multiplication: $A \cdot C = B \cdot C$
4. Division: $\frac{A}{C} = \frac{B}{C}$ provided $C \neq 0$
5. Exponent: $A^n = B^n$ provided n is an integer
6. Reciprocal: $\frac{1}{A} = \frac{1}{B}$ provided $A \neq 0, B \neq 0$
7. Means & Extremes: $\frac{C}{A} = \frac{D}{B} \Rightarrow CB = AD$ if $A \neq 0, B \neq 0$
8. Zero Product Property: $A \cdot B = 0$ if and only if
   $A = 0$ or $B = 0$

1.8 Rules for Radicals

1. Basic Definitions: $\sqrt[n]{a} \equiv a^{\frac{1}{n}}$ and $\sqrt[2]{a} \equiv \sqrt{a} \equiv a^{\frac{1}{2}}$
2. Complex Radical: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
3. Associative: $(\sqrt[n]{a})^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$
4. Simple Product: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$
5. Simple Quotient: $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
6. Complex Product: $\sqrt[n]{a}\sqrt[m]{b} = \sqrt[nm]{a^m b^n}$
7. Complex Quotient: $\frac{\sqrt[n]{a}}{\sqrt[m]{b}} = \sqrt[nm]{\frac{a^m}{b^n}}$
8. Nesting: $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$
9. Rationalization Rules for $n > m$

Numerator: $\frac{\sqrt[n]{a^m}}{b} = \frac{a}{b\sqrt[n]{a^{n-m}}}$ Denominator: $\frac{b}{\sqrt[n]{a^m}} = \frac{b\sqrt[n]{a^{n-m}}}{a}$

1.9 Order of Operations

Step 1: Perform all power raisings in the order they occur from left to right

Step 2: Perform all multiplications and divisions in the order they occur from left to right

Step 3: Perform all additions and subtractions in the order they occur from left to right

Step 4: If parentheses are present, first perform steps 1 through 3 *on an as-needed basis* within the innermost set of parentheses until a single number is achieved. Then perform steps 1 through 3 (*again, on an as-needed basis*) for the next level of parentheses until all parentheses have been systematically removed.

Step 5: If a fraction bar is present, simultaneously perform steps 1 through 4 for the numerator and denominator, treating each as totally separate problem until a single number is achieved. Once single numbers have been achieved for both the numerator and the denominator, then a final division can be performed.

1.10 Three Meanings of 'Equals'

1. **Equals** is the mathematical equivalent of the English verb "is", the fundamental verb of being. A simple but subtle use of equals in this fashion is $2 = 2$.
2. **Equals** implies an equivalency of naming in that the same underlying quantity is being named in two different ways. This can be illustrated by the expression $2003 = MMIII$. Here, the two diverse symbols on both sides of the equals sign refer to the same and exact underlying quantity.
3. **Equals** states the product (either intermediate or final) that results from a process or action. For example, in the expression $2 + 2 = 4$, we are adding two numbers on the left-hand side of the equals sign. Here, addition can be viewed as a process or action between the numbers 2 and 2. The result or product from this process or action is the single number 4, which appears on the right-hand side of the equals sign.

1.11 Rules for Logarithms

1. Definition of Logarithm to Base $b > 0$: $y = \log_b x$
   if and only if $b^y = x$
2. Logarithm of the Same Base: $\log_b b = 1$
3. Logarithm of One: $\log_b 1 = 0$
4. Logarithm of the Base to a Power: $\log_b b^p = p$
5. Base to the Logarithm: $b^{\log_b p} = p$
6. Notation for Logarithm Base 10: $Log x = \log_{10} x$
7. Notation for Logarithm Base e: $\ln x = \log_e x$
8. Product: $\log_b(MN) = \log_b N + \log_b M$
9. Quotient: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

10. Power: $\log_b N^p = p \log_b N$

11: Change of Base Formula: $\log_b N = \frac{\log_a N}{\log_a b}$

1.12 Complex Numbers

1. Properties of the imaginary unit i: $i^2 = -1 \Rightarrow i = \sqrt{-1}$

2. Definition of Complex Number: Numbers of the form
   $a + bi$ where a, b are real numbers

4. Definition of Complex Conjugate: $\overline{a + bi} = a - bi$
5. Definition of Complex Modulus: $|a + bi| = \sqrt{a^2 + b^2}$
6. Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
7. Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$

8. Multiplication:

$$(a+bi)(c+di) = ac + (ad+bc)i + bdi^2 =$$
$$ac - bd + (ad+bc)i$$

9: Division:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(\overline{c+di})}{(c+di)(\overline{c+di})} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
$$\frac{(ac+bd)+(bc-ad)i}{c^2-d^2} = \frac{ac+bd}{c^2-d^2} + \left(\frac{bc-ad}{c^2-d^2}\right)i$$

1.13 Quadratic Equations and Functions

Let $ax^2 + bx + c = 0, a \neq 0$ be a quadratic equation

1. Quadratic Formula for Solutions x: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
2. Solution Discriminator: $b^2 - 4ac$

   Two real solutions: $b^2 - 4ac > 0$
   One real solution: $b^2 - 4ac = 0$
   Two complex solutions: $b^2 - 4ac < 0$

3. Solution when $a = 0 \,\&\, b \neq 0$: $bx + c = 0 \Rightarrow x = \dfrac{-c}{b}$
4. Definition of Quadratic-in-Form Equation:
   $aw^2 + bw + c = 0$ where w is a algebraic expression
5. Definition of Quadratic Function: $f(x) = ax^2 + bx + c$
6. Axis of Symmetry for Quadratic Function: $x = \dfrac{-b}{2a}$
7. Vertex for Quadratic Function: $\left(\dfrac{-b}{2a}, \dfrac{4ac - b^2}{4a}\right)$

1.14 Cardano's Cubic Solution

Let $ax^3 + bx^2 + cx + d = 0$ be a cubic equation.

Step 1: Set $x = y - \frac{b}{3a}$

After this substitution, the above cubic becomes $y^3 + py + q = 0$ where

$$p = \left[\frac{c}{a} - \frac{b^2}{3a^2}\right] \& \; q = \left[\frac{2b^2}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right].$$

Step 2: Define u, v such that $y = u - v \;\&\; p = 3uv$

Step 3: Substitute for $y \;\&\; p$ in the equation $y^3 + py + q = 0$. This leads to

$$(u^3)^2 + qu^3 - \frac{p^3}{27} = 0$$, which is quadratic-in-form in u^3.

Step 4: Solve for $u^3 = \dfrac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}$

Step 5: Solve for $u \;\&\; v$ where $v = \frac{p}{3u}$ to obtain

$$u = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}}$$

$$v = -\sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4}{27}p^3}}{2}}$$

Step 6: Solve for x where $x = y - \frac{b}{3a} = u - v - \frac{b}{3a}$

1.15 Theory of Polynomial Equations

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$
be a polynomial written in standard form.

Eight Basic Theorems

1. Fundamental Theorem of Algebra: Every polynomial $P(x)$ of degree $N \geq 1$ has at least one solution x_0 for which $P(x_0) = 0$. This solution may be real or complex (i.e. has the form $a + bi$).

2. Numbers Theorem for Roots and Turning Points: If $P(x)$ is a polynomial of degree N , then the equation $P(x) = 0$ has up to N real solutions or *roots*. The equation $P(x) = 0$ has exactly N roots if one counts complex solutions of the form $a + bi$. Lastly, the graph of $P(x)$ will have up to $N - 1$ turning points (which includes both relative maxima and minima).

3. Real Root Theorem: If $P(x)$ is of odd degree having all real coefficients, then $P(x)$ has at least one real root.

4. Rational Root Theorem: If $P(x)$ has all integer coefficients, then any rational roots for the equation $P(x) = 0$ must have the form $\frac{p}{q}$ where p is a factor of the constant coefficient a_0 and q is a factor of the lead coefficient a_n . *Note: This result is used to form a rational-root possibility list*.

5. Complex Conjugate Pair Root Theorem: Suppose $P(x)$ has all real coefficients. If $a + bi$ is a root for $P(x)$ with $P(a + bi) = 0$, then $P(a - bi) = 0$.

6. Irrational Surd Pair Root Theorem: Suppose $P(x)$ has all rational coefficients. If $a + \sqrt{b}$ is a root for $P(x)$ with $P(a + \sqrt{b}) = 0$, then $P(a - \sqrt{b}) = 0$. *Note: the surd for the radical expression* $a \pm \sqrt{b}$ *is defined to be the quantity* $a \mp \sqrt{b}$.

7. Remainder Theorem: If $P(x)$ is divided by $(x-c)$, then the remainder R is equal to $P(c)$. *Note: this result is extensively used to evaluate a given polynomial $P(x)$ at various values of x.*

8. Factor Theorem: If c is any number with $P(c)=0$, then $(x-c)$ is a factor of $P(x)$. This means $P(x)=(x-c)\cdot Q(x)$ where $Q(x)$ is a new, reduced polynomial having degree one less than $P(x)$. The converse is also true $P(x)=(x-c)\cdot Q(x) \Rightarrow P(c)=0$.

Four Advanced Theorems

9. Root Location Theorem: Let (a,b) be an interval on the x axis with $P(a)\cdot P(b)<0$. Then there is a value $x_0 \in (a,b)$ such that $P(x_0)=0$.

10. Root Bounding Theorem: Divide $P(x)$ by $(x-d)$ to obtain $P(x)=(x-d)\cdot Q(x)+R$. Case $d>0$: If both R and all the coefficients of $Q(x)$ are positive, then $P(x)$ has no root $x_0>d$. Case $d<0$: If the roots of $Q(x)$ alternate in sign—with the remainder R "in sync" at the end—then $P(x)$ has no root $x_0<d$. *Note: Coefficients of zero can be counted either as positive or negative—which ever way helps in the subsequent determination.*

11. Descartes' Rule of Signs: Arrange $P(x)$ in standard order as shown in the title bar. The number of positive real solutions equals the number of coefficient sign variations or that number decreased by an even number. Likewise, the number of negative real solutions equals the number of coefficient sign variations in $P(-x)$ or that number decreased by an even number.

12. Turning Point Theorem: Let a polynomial $P(x)$ have degree N. Then the number of turning points for a polynomial $P(x)$ can not exceed $N-1$.

1.16 Determinants and Cramer's Rule

1. Determinant Expansions

Two by Two: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Three by Three: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

2. Cramer's Rule for a Two by Two Linear System

Given $\begin{matrix} ax + by = e \\ cx + dy = f \end{matrix}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Then $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ and $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$

4. Cramer's Rule for a Three by Three Linear System

Given $\begin{matrix} ax + by + cz = j \\ dx + ey + fz = k \\ gx + hy + iz = l \end{matrix}$ with $D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0$

Then $x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{D}, y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{D}, z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{D}$

1.17 Binomial Theorem

1. Definition of $n!$ where n is a positive integer:

$$n! = n(n-1)(n-2)...1$$

2. Special Factorials: $0! = 1$ and $1! = 1$

3. Combinatorial Symbol: $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

4. Summation Symbols:

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + a_3 + a_4 + ... + a_n$$

$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + a_{k+2} + a_{k+3} ... + a_n$$

5. Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$

6. Sum of Binomial Coefficients when $a = b = 1$: $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

7. Formula for ($r+1$) st Term: $\binom{n}{r} a^{n-r} b^r$

8. Pascal's Triangle for $n = 10$:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 256 210 120 45 10 1

1.18 Geometric Series

1. Definition: $\sum_{i=0}^{n} ar^i$ where r is the common ratio

2. Summation Formula for $\sum_{i=0}^{n} ar^i$: $\sum_{i=0}^{n} ar^i = \frac{a(1-r^{n+1})}{1-r}$

3. Summation for Infinite Number of Terms Provided $0 < r < 1$

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

1.19 Boolean Algebra

The propositions p & q are either True (T) or False (F).

1. Elementary Truth Table:

$and = \wedge : or = \vee : negation = \sim : implies = \Rightarrow, \Leftrightarrow$							
p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	F	F
F	F	T	T	T	F	T	T

2. Truth Table for Exclusive Or:

p	q	$p \overset{e}{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

3. Modus Ponens: Let $p \Rightarrow q$ and p be True.
$\therefore q$ is True

4. Chain Rule: Let $p \Rightarrow q$ and $q \Rightarrow r$.
$\therefore p \Rightarrow r$ is True.

5. Modus Tollens: Let $p \Rightarrow q$ and q be False.
$\therefore \sim q \Rightarrow \sim p$ is True

6. Fallacy of Affirming the Consequent:
Let $p \Rightarrow q$ and q be True.
$\therefore q \Rightarrow p$ is False

7. Fallacy of Denying the Antecedent:
Let $p \Rightarrow q$ and p be False.
$\therefore \sim p \Rightarrow \sim q$ is False

8. Disjunctive Syllogism for the Exclusive Or $\overset{e}{\vee}$:
Let $p \overset{e}{\vee} q$ be True and q be False.
$\therefore p$ is True

1.20 Variation Formulas

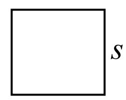
1. Direct: $y = kx$
2. Inverse: $y = \frac{k}{x}$
3. Joint: $z = kxy$
4. Inverse Joint: $z = \frac{kx}{y}$
5. Direct to Power: $y = kx^n$
6. Inverse to Power: $y = \frac{k}{x^n}$

2) Geometry

2.1 Planar Areas and Perimeters

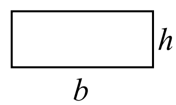
A is the planar area, P is the perimeter

1. Square:



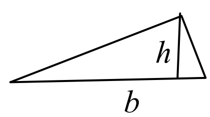
$A = s^2$ and $P = 4s$; s is the length of a side.

2. Rectangle:



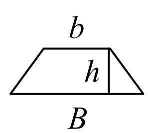
$A = bh$ and $P = 2b + 2h$; b & h are the base and height.

3. Triangle:



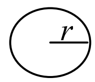
$A = \frac{1}{2}bh$; b & h are the base and altitude.

4. Trapezoid:



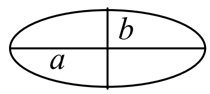
$A = \frac{1}{2}(B + b)h$; B & b are the two parallel bases, and h is the altitude.

5. Circle:



$A = \pi r^2$ and $P = 2\pi r$; r is the radius.

6. Ellipse:

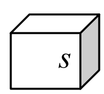


$A = \pi ab$; a & b are the half lengths of the major & minor axes.

2.2 Solid Volumes and Surface Areas

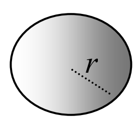
A is total surface area, V is the volume

1. Cube:



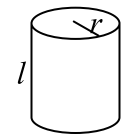
$A = 6s^2$ and $V = s^3$; s is the length of a side.

2. Sphere:



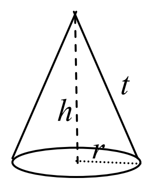
$A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$; r is the radius.

3. Cylinder:



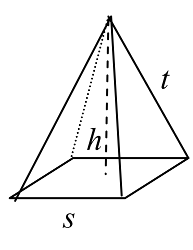
$A = 2\pi r^2 + 2\pi rl$ and $V = \pi r^2 l$;
r & l are the radius and length.

4. Cone:



$A = \pi r^2 + 2\pi rt$ and $V = \frac{1}{3}\pi r^2 h$;
r & t & h are radius, slant height, and altitude.

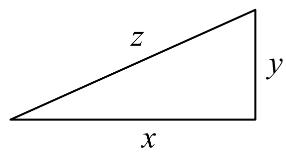
5. Pyramid (square base):



$A = s^2 + 2st$ and $V = \frac{1}{3}s^2 h$;
s & t & h are side, slant height, and altitude.

2.3 Pythagorean Theorem

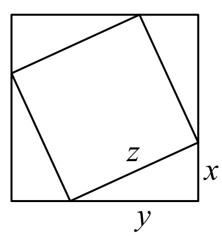
Theorem Statement: Given a right triangle with one side of length x, a second side of length y, and hypotenuse of length z.



Then: $z^2 = x^2 + y^2$

1. A Traditional Algebraic Proof: Construct a big square by bringing together four congruent right triangles where each is a replicate of the triangle shown above.

$A = (x + y)^2$



$A = z^2 + 4\left(\frac{xy}{2}\right)$

The area of the big square is given by

$A = (x + y)^2$, *or equivalently by*

$A = z^2 + 4\left(\frac{xy}{2}\right)$.

Equating:

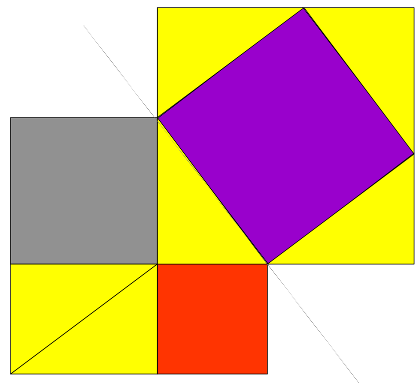
$$(x + y)^2 = z^2 + 4\left(\frac{xy}{2}\right) \Rightarrow$$

$$x^2 + 2xy + y^2 = z^2 + 2xy \Rightarrow .$$

$$x^2 + y^2 = z^2 \Rightarrow$$

$$z^2 = x^2 + y^2 \therefore$$

2. A Pre-Algebraic Visually Intuitive Proof:



3. Definition of Pythagorean Triples:

Positive integers L, M, N such that $L^2 = M^2 + N^2$

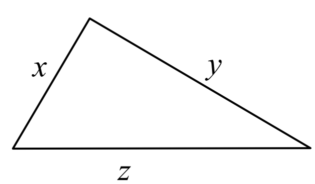
Generating formulas for Pythagorean triples:

Let $m > n > 0$ be integers.

Then $M = m^2 - n^2; N = 2mn; L = m^2 + n^2$

2.4 Heron's Formula for Triangular Area

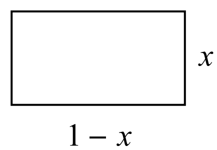
Let $p = \frac{1}{2}(x + y + z)$ be the semi-perimeter of a general triangle



Then: $A = \sqrt{p(p-x)(p-y)(p-z)}$

2.5 Golden Ratio

Definition: Let $p = 1$ be the semi-perimeter of a rectangle whose base and height are in the proportion shown. This proportion defines the Golden Ratio.



$$\frac{x}{1-x} = \frac{1-x}{1}$$

Golden Ratio

Solving: $x = 0.3819$ and $1 - x = 0.6181$

2.6 Distance and Line Formulas

Let (x_1, y_1) and (x_2, y_2) be two points where $x_2 > x_1$.

1. Distance Formula: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Midpoint Formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
3. Slope of Line: $m = \frac{y_2 - y_1}{x_2 - x_1}$
4. Point/Slope Form of Equation of Line: $y - y_1 = m(x - x_1)$
5. General Form of Equation of Line: $Ax + By + C = 0$
6. Slope/Intercept Form of Equation of Line: $y = mx + b$
7. Slope/Intercept Form; x and y Intercepts: $\frac{-b}{m}$ and b
8. Slope of Parallel Line: m
9. Slope of Line Perpendicular to a Given Line of Slope m : $\frac{-1}{m}$

2.7 Conic Section Formulas

1. General: $Ax^2 + Bxy + Cy^2 + Dx + Ey + f = 0$

2. Circle of Radius r Centered at (h,k):

   $$(x-h)^2 + (y-k)^2 = r^2$$

3. Ellipse Centered at (h,k):

   $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

   If $a > b$, the two foci are on the line $y = k$ and are given by $(h-c,k)$ & $(h+c,k)$ where $c^2 = a^2 - b^2$.

   If $b > a$, the two foci are on the line $x = h$ and are given by $(h,k-c)$ & $(h,k+c)$ where $c^2 = b^2 - a^2$.

4. Hyperbola Centered at (h,k):

   $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or}$$

   $$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

   When $\frac{(x-h)^2}{a^2}$ is to the left of the minus sign, the two foci are on the line $y = k$ and are given by $(h-c,k)$ & $(h+c,k)$ where $c^2 = a^2 + b^2$.

When $\dfrac{(y-k)^2}{b^2}$ is to the left of the minus sign,
the two foci are on the line $x = h$ and are given by
$(h, k-c)$ & $(h, k+c)$ where $c^2 = b^2 + a^2$.

5. Parabola with Vertex at (h,k) and Focal Length p :

$(y-k)^2 = 4p(x-h)$ *or*

$(x-h)^2 = 4p(y-k)$

For $(y-k)^2$, the focus is $(h+p,k)$
and the directrix is given by the line $x = h - p$.

For $(x-h)^2$, the focus is
and the directrix is given by the line $y = k - p$.

6. Transformation for removal of xy term in the general conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + f = 0.$$

First, set $\tan(2\theta) = \dfrac{B}{A-C}$ and solve for θ.

Then, let

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

3) Money and Finance

P is the amount initially borrowed or deposited.
A is the total amount gained or owed.
r is the annual interest rate.
i is the annual inflation rate.
α is an annual growth rate as in the growth rate of voluntary contributions to a fund.
r_{eff} is the effective annual interest rate.
t is the time period in years for an investment.
T is the time period in years for a loan.
N is the number of compounding periods per year.
M is the monthly payment.

3.1 Simple Interest

1. Interest alone: $I = \Pr T$
2. Total repayment over T: $R = P + \Pr T = P(1 + rT)$
3. Monthly payment over $T : M = \dfrac{P(1+rT)}{12T}$

3.2 Simple Principle Growth and Decline

1. Compounded Growth: $A = P(1 + \frac{r}{N})^{Nt}$
2. Continuous Growth: $A = Pe^{rt}$
3. Continuous Annual Inflation Rate i: $A = Pe^{-it}$

3.3 Effective Interest Rates

1. For N Compounding Periods per Year: $r_{eff} = (1 + \frac{r}{N})^{N} - 1$
2. For Continuous Interest: $r_{eff} = e^{r} - 1$

3. For a Known P, A, T : $r_{eff} = \sqrt[T]{\frac{A}{P}} - 1$

3.4 Continuous Interest IRA Growth Formulas

1. Annual Deposit D : $A = \frac{D}{r}(e^{rt} - 1)$
2. Annual Deposit D plus Initial Deposit P :

$$A = Pe^{rt} + \frac{D}{r}(e^{rt} - 1)$$

3. Annual Deposit D plus Initial Deposit P ;
   Annual Deposit Continuously Growing via $De^{\alpha t}$:

$$A = Pe^{rt} + \frac{D}{r - \alpha}(e^{rt} - e^{\alpha t})$$

4. Replacement Formula: Continuous Interest to
   Compounded Interest

   Replace e^{rt} with $(1 + \frac{r}{N})^{Nt}$

3.5 Continuous Interest Mortgage Formulas

1. First Month's Interest: $I_{1st} = \frac{rP}{12}$
2. Monthly Payment: $M = \frac{\Pr e^{rT}}{12(e^{rT} - 1)}$
3. Total Repayment ($P + I$): $A = \frac{\Pr Te^{rT}}{e^{rT} - 1}$

4. Total Interest Repayment: $I = P\left[\dfrac{rTe^{rT}}{e^{rT}-1}-1\right]$

5. Replacement Formula:
   Continuous Principle Reduction to
   Monthly Principle Reduction

   Replace e^{rT} with $(1+\frac{r}{12})^{12T}$

3.6 Continuous Interest Fixed-Rate Annuity Formula

Basic fact to remember: an annuity is a mortgage in reverse where the roles of the individual and financial institution have been interchanged. All continuous-interest mortgage formulas double as continuous-interest annuity formulas.

1. Monthly Annuity Payment: $M = \dfrac{\Pr e^{rT}}{12(e^{rT}-1)}$

3.7 Markup and Markdown

$C : Cost$

$OP : Old\,\Pr ice$

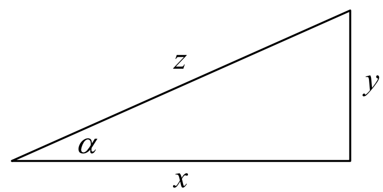
$NP : New\,\Pr ice$

$P : GivenPercent(DecimalEquivalent)$

1. Markup Based on Original Cost: $NP = (1+P)C$
2. Markup Based on Cost plus New Price: $C + P \cdot NP = NP$
3. Markup Based on Old Price: $NP = (1+P)OP$
4. Markdown Based on Old Price: $NP = (1-P)OP$
5. Percent given Old and New Price: $P = NP/OP$

4) Trigonometry

4.1 Basic Definitions: Functions & Inverses



Let the figure above be a right triangle with one side of length x, a second side of length y, and a hypotenuse of length z. The angle α is opposite the side of length. The six trigonometric functions—where each is a function of α—are defined as follows:

Arbitrary z	**For** $z = 1$	**Inverse for** $z = 1$
1. $\sin(\alpha) = \frac{y}{z}$	$\sin(\alpha) = y$	$\sin^{-1}(y) \equiv$ $\arcsin(y) = \alpha$
2. $\cos(\alpha) = \frac{x}{z}$	$\cos(\alpha) = x$	$\cos^{-1}(x) \equiv$ $\arccos(x) = \alpha$
3. $\tan(\alpha) = \frac{y}{x}$	$\tan(\alpha) = \frac{y}{x}$	$\tan^{-1}(\frac{y}{x}) \equiv$ $\arctan(\frac{y}{x}) = \alpha$
4. $\cot(\alpha) = \frac{x}{y}$	$\cot(\alpha) = \frac{x}{y}$	$\cot^{-1}(\frac{x}{y}) \equiv$ $arc\cot(\frac{x}{y}) = \alpha$

5. $\sec(\alpha) = \dfrac{z}{x}$ $\quad$ $\sec(\alpha) = \dfrac{1}{x}$ $\quad$ $\sec^{-1}(\frac{1}{x}) \equiv$ $arc\sec(\frac{1}{x}) = \alpha$

6. $\csc(\alpha) = \dfrac{z}{y}$ $\quad$ $\csc(\alpha) = \dfrac{1}{y}$ $\quad$ $\csc^{-1}(\frac{1}{y}) \equiv$ $arc\csc(\frac{1}{y}) = \alpha$

4.2 Fundamental Definition-Based Identities

1. $\csc(\alpha) = \dfrac{1}{\sin(\alpha)}$
2. $\sec(\alpha) = \dfrac{1}{\cos(\alpha)}$
3. $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$
4. $\cot(\alpha) = \dfrac{\cos(\alpha)}{\sin(\alpha)}$
5. $\tan(\alpha) = \dfrac{1}{\cot(\alpha)}$

4.3 Pythagorean Identities

1. $\sin^2(\alpha) + \cos^2(\alpha) = 1$
2. $1 + \tan^2(\alpha) = \sec^2(\alpha)$
3. $1 + \cot^2(\alpha) = \csc^2(\alpha)$

4.4 Negative Angle Identities

1. $\sin(-\alpha) = -\sin(\alpha)$

2. $\cos(-\alpha) = \cos(\alpha)$
3. $\tan(-\alpha) = -\tan(\alpha)$
4. $\cot(-\alpha) = -\cot(\alpha)$

4.5 Sum and Difference Identities

1. $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
2. $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
3. $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
4. $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
5. $\tan(\alpha+\beta) = \dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$
6. $\tan(\alpha-\beta) = \dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

4.6 Double Angle Identities

1. $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
2. $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

   $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
3. $\tan(2\alpha) = \dfrac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

4.7 Half Angle Identities

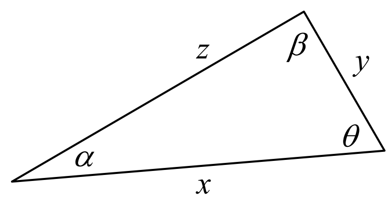
1. $\sin(\dfrac{\alpha}{2}) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{2}}$

2. $\cos(\frac{\alpha}{2}) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$

3. $\tan(\frac{\alpha}{2}) = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}} = \frac{\sin(\alpha)}{1+\cos(\alpha)} = \frac{1-\cos(\alpha)}{\sin(\alpha)}$

4.8 General Triangle Formulas

Applicable to all triangles: right and non-right



1. Sum of Interior Angles: $\alpha + \beta + \theta = 180^0$
2. Law of Sines: $\frac{y}{\sin(\alpha)} = \frac{x}{\sin(\beta)} = \frac{z}{\sin(\theta)}$
3. Law of Cosines:

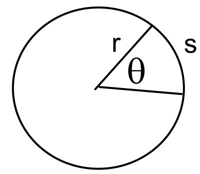
   a) $y^2 = x^2 + z^2 - 2xz\cos(\alpha)$
   b) $x^2 = y^2 + z^2 - 2yz\cos(\beta)$
   c) $z^2 = x^2 + y^2 - 2xy\cos(\theta)$

4. Area Formulas for a General Triangle:

   a) $A = \frac{1}{2}xz\sin(\alpha)$
   b) $A = \frac{1}{2}yz\sin(\beta)$
   c) $A = \frac{1}{2}xy\sin(\theta)$

4.9 Arc and Sector Formulas

1. Arc Length s : $s = r\theta$
2. Area of a Sector: $A = \frac{1}{2} r^2 \theta$



4.10 Degree/Radian Relationship

1. Basic Conversion: $180^0 = \pi$ radians
2. Conversion Formulas

From	To	Multiply by
Radians	Degrees	$\frac{180^0}{\pi}$
Degrees	Radians	$\frac{\pi}{180}$

4.11 Addition of Sine and Cosine

1. $a\sin\theta + b\cos\theta = k\sin(\theta + \alpha)$ where

$$k = \sqrt{a^2 + b^2}$$

$$\alpha = \sin^{-1}\left[\frac{b}{\sqrt{a^2 + b^2}}\right]$$

or

$$\alpha = \cos^{-1}\left[\frac{a}{\sqrt{a^2 + b^2}}\right]$$

4.12 Polar Form of Complex Numbers

1. $a+bi = r(\cos\theta + i\sin\theta)$ where

$$r = \sqrt{a^2+b^2}\ , \theta = Tan^{-1}\left[\frac{b}{a}\right]$$

2. Definition of $re^{i\theta}$: $re^{i\theta} = r(\cos\theta + i\sin\theta)$

3. Euler's Famous Equality: $e^{i\pi} = -1$

4. Traditional Statement of de Moivre's Theorem:

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos[n\theta] + i\sin[n\theta])$$

5. Alternate Statement of de Moivre's Theorem:

$$(re^{i\theta})^n = r^n e^{in\theta}$$

6. Polar Form Multiplication: $r_1 e^{i\alpha} \cdot r_2 e^{i\beta} = r_1 \cdot r_2 e^{i(\alpha+\beta)}$

7. Polar Form Division: $\dfrac{r_1 e^{i\alpha}}{r_2 e^{i\beta}} = \dfrac{r_1}{r_2} e^{i(\alpha-\beta)}$

4.13 Rectangular to Polar Coordinates

$$(x, y) \Leftrightarrow (r, \theta)$$
$$x = r\cos\theta, y = r\sin\theta$$
$$r = \sqrt{x^2+y^2}, \theta = \tan^{-1}(x/y)$$

5) Elementary Calculus

5.1 Basic Differentiation Rules

1. Limit Definition of: $f'(x) = \lim_{h \to 0} \left[\dfrac{f(x+h) - f(x)}{h} \right]$

2. Constant: $[k]' = 0$

3. Power: $[x^n]' = nx^{n-1}$, n can be <u>any exponent</u>

4. Coefficient: $[af(x)]' = af'(x)$

5. Sum/Difference: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

6. Product: $[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$

7. Quotient: $\left[\dfrac{f(x)}{g(x)} \right]' = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

8. Chain: $[f(g(x))]' = f'(g(x))g'(x)$.

9. Inverse: $[f^{-1}(x)]' = \dfrac{1}{f'(f^{-1}(x))}$

10. Generalized Power: $\left[\{f(x)\}^n \right]' = n\{f(x)\}^{n-1} f'(x)$;

    Again, n can be <u>any exponent</u>

5.2 Transcendental Differentiation

1. $[\ln x]' = \dfrac{1}{x}$

2. $[\log_a x]' = \dfrac{1}{x \ln a}$

3. $[e^x]' = e^x$

4. $[a^x]' = a^x \ln a$

5. $[\sin x]' = \cos x$

6. $[\sin^{-1}(x)]' = \dfrac{1}{\sqrt{1-x^2}}$

7. $[\cos x]' = -\sin x$

8. $[\cos^{-1}(x)]' = \dfrac{-1}{\sqrt{1-x^2}}$

9. $[\tan x]' = \sec^2 x$

10. $[\tan^{-1}(x)]' = \dfrac{1}{1+x^2}$

11. $[\sec x]' = \sec x \tan x$

12. $[\sec^{-1}(x)]' = \dfrac{1}{|x|\sqrt{x^2-1}}$

5.3 Basic Antidifferentiation Rules

1. Constant: $\int k dx = kx + C$

2. Coefficient: $\int af(x)dx = a\int f(x)dx$

3. Power:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C, n = -1$$

4.Sum/Difference: $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

5. Parts: $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

6. Chain: $\int f'(g(x))g'(x)dx = f(g(x)) + C$

7. Generalized Power:

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C, n = -1$$

5.4 Transcendental Antidifferentiation

1. $\int \ln x dx = x \ln x - x + C$

2. $\int e^x dx = e^x + C$

3. $\int a^x dx = \frac{a^x}{\ln a} + C$

5. $\int \cos x dx = \sin x + C$

6. $\int \sin x dx = -\cos x + C$

7. $\int \tan x dx = \ln|\cos x| + C$

8. $\int \sec x dx = \ln|\sec x + \tan x| + C$

9. $\int \sec x \tan x dx = \sec x + C$

10. $\int \sec^2 x dx = \tan x + C$

11. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(\frac{x}{a}) + C$

12. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$

13. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

5.5 Lines and Approximation

1. Tangent Line at $(a, f(a))$: $y - f(a) = f'(a)(x-a)$
2. Normal Line at $(a, f(a))$: $y - f(a) = \dfrac{-1}{f'(a)}(x-a)$
3. Linear Approximation: $f(x) \cong f(a) + f'(a)(x-a)$
4. Second Order Approximation:
$$f(x) \cong f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$
5. Newton's Iterative Root-Approximation Formula:
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

5.6 The Fundamental Theorem of Calculus

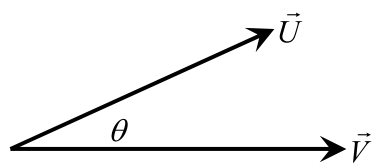
Consider the definite integral $\int_a^b f(x)dx$, which can be thought of as a continuous addition process on the interval $[a,b]$, a process that sums millions upon millions of tiny quantities having the general form $f(x)dx$ from $x = a$ to $x = b$. Now, let $F(x)$ be any antiderivative for $f(x)$ where, by definition, we have that $F'(x) = f(x)$. Then, the summation process $\int_a^b f(x)dx$ can be evaluated by the alternative process

$$\int_a^b f(x)dx = F(x)\,|_a^b = F(b) - F(a).$$

6) Elementary Vector Algebra

6.1 Basic Definitions and Properties

Let $\vec{V} =< v_1, v_2, v_3 >, \vec{U} =< u_1, u_2, u_3 >$



1. $\vec{U} \pm \vec{V} =< u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3 >$
2. $(\alpha)\vec{U} =< \alpha u_1, \alpha u_2, \alpha u_3 >$
3. $-\vec{U} = (-1)\vec{U}$
4. $\vec{0} = (0,0,0)$
5. $[\vec{U}] = \sqrt{u_1^2 + u_2^2 + u_3^2}$
6. Unit Vector Parallel to $\vec{V} : \frac{1}{[\vec{V}]}\vec{V}$
7. Two Parallel Vectors: $\vec{V} \parallel \vec{U} \Rightarrow \exists c, st \vec{V} = (c)\vec{U}$

6.2 Dot Products

1. Definition of Dot Product: $\vec{U} \bullet \vec{V} = u_1 v_1 + u_2 v_2 + u_3 v_3$
2. Angle $\theta : \cos\theta = \frac{\vec{U} \bullet \vec{V}}{[\vec{U}][\vec{V}]}$

3. Orthogonal Vectors: $\vec{U} \bullet \vec{V} = 0$

4. Projection of $\vec{U}$ onto $\vec{V}$:

$$proj_{\vec{V}}(\vec{U}) = \left[\frac{\vec{U} \bullet \vec{V}}{[\vec{V}]^2}\right]\vec{V} = \left[\frac{\vec{U} \bullet \vec{V}}{[\vec{V}]}\right]\frac{\vec{V}}{[\vec{V}]} =$$

$$\left[[\vec{U}]\cos\theta\right]\frac{\vec{V}}{[\vec{V}]}$$

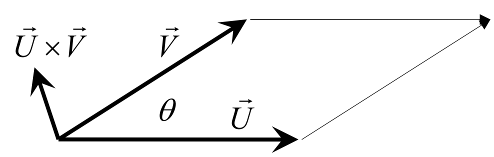
6.3 Cross Products

1. Definition of Cross Product: $\vec{U} \times \vec{V} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

2. Orientation of $\vec{U} \times \vec{V}$; Orthogonal to Both $\vec{U}$ and $\vec{V}$:

$$\vec{U} \bullet (\vec{U} \times \vec{V}) = \vec{V} \bullet (\vec{U} \times \vec{V}) = 0$$

3. Area of Parallelogram

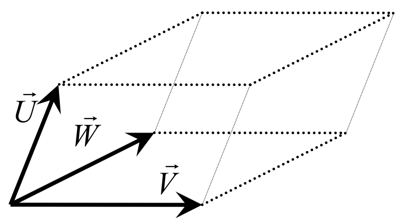


$$[\vec{U} \times \vec{V}] = [\vec{U}][\vec{V}]\sin\theta$$

4. Interpretation of the Triple Scalar Product:

$$\vec{U} \bullet (\vec{V} \times \vec{W}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The triple scalar product is numerically equal to the volume of the parallelepiped below.



6.4 Line and Plane Equations

1. Line parallel to $< a, b, c >$ and
   Passing Through (x_1, y_1, z_1) .

   If (x, y, z) is a point on the line, then:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

2. Equation of Plane Normal to $< a, b, c >$ and
   Passing Through (x_1, y_1, z_1) .

   If (x, y, z) is a point on the plane, then:

$$< a, b, c > \bullet < x - x_1, y - y_1, z - z_1 > = 0 .$$

3. Distance D between a point & plane:

If a point is given by (x_0, y_0, z_0)
and $ax + by + cz + d = 0$ is a plane, then

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

6.5 Miscellaneous Vector Equations

1. The Three Direction Cosines:

$$\cos\alpha = \frac{v_1}{[\vec{V}]}, \cos\beta = \frac{v_2}{[\vec{V}]}, \cos\gamma = \frac{v_3}{[\vec{V}]},$$

2. Definition of Work: constant force $\vec{F}$ along the path $P\vec{Q}$

$$W = \vec{F} \bullet P\vec{Q} = [proj_{P\vec{Q}}(\vec{F})][P\vec{Q}]$$

7) Probability and Statistics

7.1 Probability Formulas

Let U be a universal set consisting of all possible events.
Let Φ be the empty set consisting of no event.
Let $A, B \subset U$

1. Basic Formula: $P = \dfrac{favorable - number - of - ways}{total - number - of - ways}$

2. Fundamental Properties: $$P(U) = 1$$ $$P(\Phi) = 0$$

3. Order Relationship: $A \subset U \Rightarrow 0 \le P(A) \le 1$

4. Complement Law: $P(A) = 1 - P(\sim A)$

5. Addition Law: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

6. Conditional Probability Law: $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$ $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

7. General Multiplication Law: $$P(A \cap B) = P(B) \cdot P(A \mid B)$$ $$P(A \cap B) = P(A) \cdot P(B \mid A)$$

8. Independent Events (IE): $A \cap B = \Phi$

9. Multiplication Law for IE: $P(A \cap B) = P(A) \cdot P(B)$

7.2 Basic Concepts of Statistics

1. A **set** is an aggregate of individual items—animate or inanimate.
2. An **element** of the set is a particular item in the set.
3. An **observation** associated with the element is any attribute of interest.
4. A **statistic** associated with the element is any measurement of interest. Any statistic is an observation, but not all observations are statistics.
5. **Statistics**: the science of drawing conclusions from the totality of observations.
6. A **population** is the totality of elements that one wishes to study by making observations.
7. A **sample** is that population subset that one has the resources to study.
8. **Random sample**: where all population elements have equal probability of access.

Let a set consist of N elements where there has been observed one statistic of a similar nature for each element. The data set of all observed statistics is denoted by $\{x_1, x_2, x_3, \ldots, x_N\}$. The corresponding rank-ordered data set is a re-listing of the individual statistics $\{x_1, x_2, x_3, \ldots, x_N\}$ in numerical order from smallest to largest. Data sets can come from either populations or from samples. More than likely, the data set will be considered a sample and will be utilized to make predictions about a corresponding and much larger population.

7.3 Measures of Central Tendency

1. Sample Mean or Average $\bar{x}$: $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$
2. Population Mean or Average μ : $\mu = \frac{1}{N} \sum_{i=1}^{N} x_i$

3. Median $\tilde{x}$: the middle value in a rank-ordered data set

Calculation Procedure: The median $\tilde{x}$ is the actual middle statistic if there is an odd number of data points. The median $\tilde{x}$ is the average of the two middle statistics if there is an even number of data points.

4. Mode M : the data value or statistic that occurs most often.

5. Multi-Modal Data Set: a data set with two or more modes

7.4 Measures of Dispersion

1. Range $R : R = x_L - x_S$ where x_L is the largest data value in the data set and x_S is the smallest data value

2. Sample Standard Deviation s : $s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$.

3. Population Standard Deviation σ : $\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$.

4. Definition of Sample Variance: s^2

5. Definition of Population Variance: σ^2

6. Sample Coefficient of Variation $C_{VS} : C_{VS} = \frac{s}{\bar{x}}$

7. Population Coefficient of Variation C_{VP} : $C_{VP} = \frac{\sigma}{\mu}$

8. Z-Score for a Given Data Value x_i : $z_i = \frac{x_i - \bar{x}}{s}$

7.5 Sampling Distribution of the Mean

The mean $\bar{x}$ is formed from a sample of individual data points randomly selected from either an infinite or finite population. The number of data points selected is given by n. The sample is considered a Large Sample if $n \geq 30$; a Small Sample if $n < 30$.

1. Expected Value of $\bar{x}$: $E(\bar{x}) = \mu$
2. Standard Deviation of $\bar{x}$:

Infinite Population	Finite Population of Count N
$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$	$\sigma_{\bar{x}} = \sqrt{\dfrac{N-n}{N-1}}\dfrac{\sigma}{\sqrt{n}}$

3. Large Sample Z-score for $\bar{x}_i$: $z_i = \dfrac{\bar{x}_i - \mu}{\sigma / \sqrt{n}}$

   When σ is unknown, substitute s.

4. Interval Estimate of Population Mean:

Large-Sample Case	Small-Sample Case
$\bar{x} \pm z_{\frac{\alpha}{2}} \cdot \left[\dfrac{\sigma}{\sqrt{n}}\right]$	$\bar{x} \pm t_{\frac{\alpha}{2}} \cdot \left[\dfrac{s}{\sqrt{n}}\right]$

No assumption about the underlying population needs to be made in the large-sample case. In the small-sample case, the underlying population is assumed to be normal or nearly so. When σ is unknown in the large-sample case, substitute s.

5. Sampling Error E: $E = z_{\frac{\alpha}{2}} \cdot \left[\dfrac{\sigma}{\sqrt{n}}\right]$

6. Sample Size Needed for a Given Error E: $n = \left[\dfrac{z_{\frac{\alpha}{2}} \cdot \sigma}{E}\right]^2$.

7.6 Sampling Distribution of the Proportion

The proportion p is a quantity formed from a sample of individual data points randomly selected from either an infinite or finite population. The proportion can be thought of as a mean formulated from a sample where all the individual values are either zero (0) or one (1). The number of data points selected is given by n. The sample is considered a Large Sample if both $np \geq 5$ and $n(1-p) \geq 5$.

1. Expected Value of $\overline{p}$: $E(\overline{p}) = \mu$
2. Standard Deviation of $\overline{p}$:

Infinite Population	Finite Population of Count N
$\sigma_{\overline{p}} = \sqrt{\dfrac{p(1-p)}{n}}$	$\sigma_{\overline{p}} = \sqrt{\dfrac{N-n}{N-1}}\sqrt{\dfrac{p(1-p)}{n}}$

3. Interval Estimate of Population Proportion:

$$\overline{p} \pm z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$$

Use $\overline{p} = .5$ in $\sqrt{\dfrac{\overline{p}(1-\overline{p})}{n}}$ if clueless on the initial size of $\overline{p}$.

4. Sampling Error E: $E = z_{\frac{\alpha}{2}} \cdot \sqrt{\dfrac{p(1-p)}{n}}$

5. Sample Size Needed for a Given Error E: $n = \dfrac{z_{\frac{\alpha}{2}}^{\;2} \cdot p(1-p)}{E^2}$

Worse case for above with proportion unknown: $n = \dfrac{z_{\frac{\alpha}{2}}^{\;2}}{4E^2}$